Defining ideals in numerical semigroup rings with arithmetic pseudo-Frobenius numbers

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Numerical semigroups and their semigroup rings provide an important class of rings in commutative algebra. A fundamental problem on numerical semigroup rings is to compute generators of their defining ideals. As an example, Herzog [2] computed the defining ideal of a numerical semigroup ring with embedding dimension three. However, when the embedding dimension is four or larger, the structure of the defining ideal is not fully understood. Recently the following interesting conjecture was posed by Cuong, Kien, Truong and, Matsuoka.

To introduce the conjecture, let us fix some notation. Let k be a field and H the numerical semigroup minimally generated by $a_0, a_1, \ldots, a_{n-1}$ with $a_0 = \min(H \setminus \{0\})$. We put $S = k[X_0, X_1, \ldots, X_{n-1}]$ the polynomial ring over k with grading $\deg X_i = a_i$, and $\varphi : S \to k[H]$ the graded ring homomorphism defined by $\varphi(X_i) = t^{a_i}$ for each $0 \le i \le n-1$. Let $V = k[t]$ be the polynomial ring over k. The numerical semigroup ring, denoted by $k[H]$, is defined as

$$k[H] := k[t^{a_0}, t^{a_1}, \ldots, t^{a_{n-1}}] \subseteq k[t].$$

The kernel of homomorphism φ is denoted by I_H. For a matrix M whose entries are in S, $\mathrm{I}_2(M)$ denotes the ideal of S generated by 2-minors of M. An integer $p \in \mathbb{Z} \setminus H$ is called a pseudo-Frobenius number if $p + h \in H$ for all $h \in H$. Let $\mathrm{PF}(H)$ be a set of all pseudo-Frobenius numbers of H.

Conjecture. (*Cuong-Kien-Truong-Matsuoka [4, Conjecture 1.1]*) *With the notation above, the following conditions are equivalent.*

(1) $I_H = \mathrm{I}_2\begin{pmatrix} X_0^{l_0} & X_1^{l_1} & \cdots & X_{n-2}^{l_{n-2}} & X_{n-1}^{l_{n-1}} \\ X_1^{m_1} & X_2^{m_2} & \cdots & X_{n-1}^{m_{n-1}} & X_0^{m_0} \end{pmatrix}$ *for some integers* $l_0, l_1, \ldots, l_{n-1}, m_0, m_1, \ldots, m_{n-1} > 0$, *after suitable permutations on* $a_0, a_1, \ldots, a_{n-1}$.

(2) The set $\mathrm{PF}(H)$ *forms an arithmetic sequence of length* $n-1$.

The implication from (1) to (2) has already been established in [3]. On the other hand, the implication from (2) to (1) remains open in general, but has been verified in certain special cases, including almost symmetric semigroups [1], semigroups with maximal embedding dimension [3], generalized repunit numerical semigroups [5] and, stretched numerical semigroup rings [4].

We prove that this conjecture holds when the embedding dimension n of H is not small comparing its multiplicity a_0.

Theorem. *Conjecture holds when* $a_0/2 + 1 \le n$.

References

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